AoPS Online
Art of Problem Solving

Students enrolling in **Scholars Math 11.1: Group Theory** should be comfortable with arithmetic mod n, the complex numbers (polynomials, DeMoivre's Theorem, and roots of unity in particular), advanced counting techniques, and also be very strong in mathematical arguments and problem solving.

Examples are given below. Modular arithmetic and complex numbers will be used extensively as examples in this class and you should be fluent with these before trying this class. We will also use counting arguments repeatedly and you should be comfortable with these types of computations.

Modular arithmetic

1. Solve the following congruences:
   (a) $3x \equiv 5 \pmod{11}$
   (b) $6x - 4 \equiv x + 9 \pmod{74}$
   (c) $3x \equiv 5 \pmod{42}$
   (d) $5x + 1 \equiv 2x + 7 \pmod{21}$
2. Find integers a and b such the following equation is true for all x:
$$x^2 + x + 1 \equiv (x-a)(x-b) \pmod{7}.$$
3. Find the units digit of $7^{(7^7)}$.

Complex numbers

4. Compute $\dfrac{i+1}{i-1}$.
5. Compute $(1+i\sqrt{3})^{23}$.
6. The complex number $1+i$ is a root of the cubic $x^3 - x^2 + 2$. Find the other two roots.
7. Find the roots of $x^6 + x^4 + x^2 + 1$.
8. Let z be a complex number such that $z^2 - z + 1 = 0$. Compute $z + z^5$.

Counting

9. How many ways are there to color three sides of a cube red and the other three sides black, calling two colorings the same if we can rotate one to the other?
10. How many ways are there to place one red and 4 (indistinguishable) blue balls into 5 indistinguishable boxes? (We allow any number of balls, including zero, in each box.)
11. We wish to color each of the integers $1, 2, 3, \ldots, 10$ red, green, or blue, so that no two numbers a and b with $a - b$ odd have the same color. (We do not require that all three colors be used.) In how many ways can this be done? (This problem appeared at the Harvard-MIT Mathematics Tournament.)

12. The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair 15. Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N(5!)^3$. Find N. (This problem originally appeared on the AIME.)

Don't look at the next page until you've attempted all the problems!

The answers are below.

1. (a) $x \equiv 9 \pmod{11}$
   (b) $x \equiv 47 \pmod{74}$
   (c) This has no solution
   (d) $x \equiv 2, 9,$ or $16 \pmod{21}$.
2. The congruence classes of 2 and 4 are the only zeroes of this polynomial, so $x^2 + x + 1 \equiv (x-a)(x-b) \equiv (x-2)(x-4) \pmod{7}$.
3. 3
4. $-i$
5. $2^{22}(1 - i\sqrt{3})$
6. $1 - i$, -1
7. $\dfrac{\pm 1 \pm i}{\sqrt{2}}$, $\pm i$
8. 1
9. 2
10. 12
11. 186
12. 346