If you can solve nearly all of the following problems with little difficulty, then the online class **Scholars Math 9.1: Intermediate Counting & Probability** would only serve as a review for you.

1. You have two bags. Bag A has 3 white marbles and 2 black ones; Bag B has 7 white marbles and 1 black one. A bag is chosen at random and a ball randomly removed from it. Given that the ball is white, what is the probability that the chosen bag was Bag A?

2. Oliver Rock is certain that the lottery is rigged. He's made tons of lists and carefully tracked lottery results and he's sure. His calculations show that at least 90% of the time there shouldn't be a single pair of consecutive numbers among the winning numbers, which are 6 integers randomly chosen from 1 to 44, inclusive. But over the last 5 years, only around 46% of the time has there not been a pair of consecutive numbers. Figure out whether or not Oliver is right by finding the probability that a drawing of 6 different numbers out of the numbers 1 through 44 will not have a single pair of consecutive numbers. (The order the numbers are chosen doesn't matter.)

3. Find a closed-form expression equivalent to

$$\sum_{j=0}^{n} \frac{\binom{n}{j}}{n^j(j+1)}.$$

4. Six namecards are at the six seats at a table. The six people whose names are on the cards sit at the table without looking at the cards. How many ways can they sit so that exactly one of them is sitting at a seat with his or her own name on the namecard?

5. Only 0.01% of people have triskaidekaphobia. The Dreizehn Club has developed a test for the phobia. If you have Triskadekaphobia, the test is 99% likely to identify that you have the disease. Unfortunately, it is also 5% likely to claim you have the disease when you don't. If you take the test twice and it twice claims you have the disease, what is the probability that you do have the disease?

6. Brazil defeats Germany in a wild World Cup Final by the score of 8 to 6. Assuming the 14 goals could be equally likely scored in any order, what is the probability that the score was never tied after the first goal?

7. Find the number of solutions in nonnegative integer triplets (x, y, z) to each of the following.
   (a) $x + y + z = 14.$
   (b) $x + y + z \leq 23.$
   (c) $2x + y + z \leq 17.$

8. Six people in chemistry class are going to be placed in three groups of two. The teacher asks each student to write down the name of that student's desired partner. If each student picks a favorite partner at random, what is the probability that there is at least one pair of students who have picked each other?

9. A partition of an integer n is a collection of positive integers whose sum is n. For example, the five partitions of 4 are: $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, 4. Prove that the number of partitions of an integer into distinct numbers (such as $1+3$ and 4 in the list above) equals the number of partitions into odd numbers only (such as $1+1+1+1$ and $1+3$ in the list above).

10. Prove that

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{1}{2}\right)^k = 2^n.$$

Don't look at the next page until you've attempted all the problems!

The answers are below. (The answers to problem sets and challenges given in the class will include full detailed solutions as opposed to the mere answers provided below.)

1. $24/59$
2. The probability of drawing the numbers without a pair of consecutive numbers is $35853/77572$, which is about 46%.
3. $\left(\frac{n+1}{n}\right)^n - \frac{n}{n+1}$
4. 264
5. $99/2624$
6. $1/7$
7. (a) 120

   (b) 2600

   (c) 615
8. $1653/3125$
9. The generating function for the number of partitions into distinct integers is $(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$, while the generating function for the number of partitions into odd integers is $(1+x+x^2+x^3+\cdots)(1+x^3+x^6+x^9+\cdots)(1+x^5+x^{10}+\cdots)\cdots = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$. Since $[(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots][(1-x)(1-x^3)(1-x^5)\cdots] = 1$, these two generating functions are the same.
10. Consider the generating function

$$\frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$

Setting $x = 1/2$ produces the desired relationship.