If you can solve nearly all of the following problems with little difficulty, then the **Scholars Math 9.2: Intermediate Number Theory** course would only serve as a review for you.

1. Compute the largest prime factor of
$$3(3(3(3(3(3(3(3(3(3(3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1)+1.$$
2. Assuming the expression converges, determine the largest integer n with $n \leq 9,000,000$ for which $\sqrt{n+\sqrt{n+\sqrt{n+\ldots}}}$ is rational.
3. The sequence $1, 3, 5, 7, 9, 15, 17, \ldots$ consists of the positive integers whose binary representation reads the same left to right as right to left. (Such as 9, which is written as 1001_2 in binary.) Find the number of 1's in the binary representation of the 1717^{th} term of the sequence.
4. What is the largest positive integer n for which n^3+100 is divisible by $n+10$?
5. Let m and n be positive integers with $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively to the last three digits of 1978^n. Find m and n such that $m+n$ has its least value.
6. Find all positive integers a, n such that
$$a^{n+1}-(a+1)^n = 2001.$$
7. Find with proof all primes that are one more than the square of a Fibonacci number.
8. Explain how to generate all ordered pairs of positive integers (x, y) to the Diophantine equation
$$x^2-2y^2=1.$$

Don't look at the next page until you've attempted all the problems!

The answers are below.

1. 73
2. 8,997,000
3. 12
4. 890
5. $m = 3$, $n = 103$
6. (13, 2) is the only solution.
7. 2 and 5. Proof on next page.
8. For a non-negative integer n, let $(3 + 2\sqrt{2})^n = a + b\sqrt{2}$, where a and b are positive integers, then (a, b) is a solution and all solutions can be generated in this way.

Proof for problem 7:

The only such primes are $\boxed{2}$ and $\boxed{5}$.

Claim: For all $n \geq 2$,

$$F_n^2 + 1 = \begin{cases} F_{n-1}F_{n+1} & \text{if } n \text{ is even,} \\ F_{n-2}F_{n+2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Claim: We prove by induction, with a separate induction for odd n and even n. For the base cases, note that

$$\begin{aligned} F_2^2 + 1 &= 2 = 1 \cdot 2 = F_1F_3 && (1) \\ F_3^2 + 1 &= 5 = 1 \cdot 5 = F_1F_5 && (2) \end{aligned}$$

Assume for even n that

$$F_n^2 + 1 = F_{n-1}F_{n+1}.$$

Now, add $F_{n+1}^2 + 2F_nF_{n+1}$ to both sides:

$$\begin{aligned} F_{n+1}^2 + 2F_nF_{n+1} + F_n^2 + 1 &= F_{n-1}F_{n+1} + F_{n+1}^2 + 2F_nF_{n+1} \\ (F_{n+1} + F_n)^2 + 1 &= F_{n+1}(F_{n-1} + 2F_n + F_{n+1}) \\ F_{n+2}^2 + 1 &= F_{n+1}[(F_{n-1} + F_n) + (F_n + F_{n+1})] \\ F_{n+2}^2 + 1 &= F_{n+1}(F_{n+1} + F_{n+2}) \\ F_{n+2}^2 + 1 &= F_{n+1}F_{n+3} \end{aligned}$$

and our first induction is complete.

Assume for odd n that

$$F_n^2 + 1 = F_{n-2}F_{n+2}.$$

Now, add $F_{n+1}^2 + 2F_nF_{n+1}$ to both sides:

$$\begin{aligned} F_{n+1}^2 + 2F_nF_{n+1} + F_n^2 + 1 &= F_{n-2}F_{n+2} + F_{n+1}^2 + 2F_nF_{n+1} \\ (F_{n+1} + F_n)^2 + 1 &= (F_n - F_{n-1})F_{n+2} + F_{n+1}(F_{n+1} + 2F_n) \\ F_{n+2}^2 + 1 &= [F_n - (F_{n+1} - F_n)]F_{n+2} + F_{n+1}[(F_{n+1} + F_n) + F_n] \\ F_{n+2}^2 + 1 &= (2F_n - F_{n+1})F_{n+2} + F_{n+1}(F_{n+2} + F_n) \\ F_{n+2}^2 + 1 &= 2F_nF_{n+2} - F_{n+1}F_{n+2} + F_{n+1}F_{n+2} + F_nF_{n+1} \\ F_{n+2}^2 + 1 &= 2F_nF_{n+2} + F_nF_{n+1} \\ F_{n+2}^2 + 1 &= F_n(2F_{n+2} + F_{n+1}) \\ F_{n+2}^2 + 1 &= F_n[F_{n+2} + (F_{n+1} + F_{n+2})] \\ F_{n+2}^2 + 1 &= F_n(F_{n+2} + F_{n+3}) \\ F_{n+2}^2 + 1 &= F_nF_{n+4} \end{aligned}$$

and our second induction is complete.

*Note that we could also combine the inductions compactly since we are adding the same quantity to both sides in both cases.

Now that we have established induction we note that F_n^2+1 is always the product of two other Fibonacci numbers. Such a product can be prime only when the smaller of the two Fibonacci numbers is 1.

For even n, $F_n^2+1=F_{n-1}F_{n+1} \Rightarrow F_{n-1}=1$. $F_1=F_2=1$, but since n is even $n=2$, so $F_n^2+1=F_2^2+1=1^2+1=2$.

For odd n, $F_n^2+1=F_{n-2}F_{n+2} \Rightarrow F_{n-2}=1$. $F_1=F_2=1$, but since n is odd $n=3$, so $F_n^2+1=F_3^2+1=2^2+1=5$.

Both 2 and 5 are prime and we conclude that they are the only two such primes.