If you can solve nearly all of the following problems with little difficulty, then you are ready for **Scholars Math 9.2: Intermediate Number Theory**.

1. (a) Find the greatest common divisor and the least common multiple of 240 and 840.
   (b) How many positive factors does 840 have?
   (c) What is the product of the divisors of 240? (Answer in exponential form.)
   (d) My backyard is a rectangle with area 480 square feet and its length and width are both an integer number of feet. If both dimensions are greater than 7 feet and the length is longer than the width, how many possible lengths are there for my backyard?
2. We use subscripts to denote what base a number is in.
   (a) Write 321_5 in base 8.
   (b) Add $4216_7 + 11366_7$. Keep your answer in base 7.
   (c) If d is a digit in the number $4d763_8$ which is a multiple of 7, what is the value of the digit d?
3. Solve for a:
   (a) $a + 3 \equiv 1 \pmod 5$
   (b) $a + 2 \equiv 4 \pmod 7$ and $a - 2 \equiv 4 \pmod 9$
   (c) $4a + 4 \equiv 3 \pmod 6$
4. What are the possible choices for the digits A and B if the number $2A1B6$ is divisible by both 9 and 4?
5. What is the units digit of $32^{21}49^{98}$?
6. A certain integer n has 40 positive divisors including 1 and n. What is the largest number of primes that could divide n?
7. How many of the positive factors of 720 are also divisible by 12?
8. How many natural numbers are divisors of exactly 2 of the 3 integers 240, 600, and 750?
9. If 14 times a natural number has a units digit of 2 when expressed in base 12, what is the units digit of the natural number when expressed in base 6?
10. Prove that every perfect square has an odd number of positive factors and that every nonsquare has an even number of positive factors.
11. How many of the first 1000 natural numbers have a remainder of 1 when divided by 5, a remainder of 2 when divided by 6, and a remainder of 5 when divided by 7?

Don't look at the next page until you've attempted all the problems!

The answers are below.

1. (a) gcd$(240, 840) = 120$; lcm$[240, 840] = 1680$
   (b) 32
   (c) 240^{10}
   (d) 6
2. (a) 126_8
   (b) 15615_7
   (c) 1
3. (a) $a \equiv 3 \pmod 5$
   (b) $a \equiv 51 \pmod{63}$
   (c) no solutions
4. $(A, B) = (0, 9)$; $(2, 7)$; $(4, 5)$; $(6, 3)$; $(8, 1)$; $(9, 9)$
5. 2
6. 4
7. 12
8. 12
9. 1
10. The prime factorization of a perfect square has only even exponents. As an example, we have $144 = 2^4 \times 3^2$. Therefore, the number of positive factors, as the product of these exponents each increased by one, must be odd. Continuing our earlier example, the number of factors of 144 is $(4 + 1)(2 + 1) = 15$.

    Similarly, a nonsquare must have an odd exponent among the exponents of its prime factorization. For example, $80 = 2^4 \times 5^1$. Thus, increasing these exponents by one will produce at least one even number, so the ensuing product of these incremented exponents must be even. Again continuing our example, the number of factors of 80 is $(4 + 1)(1 + 1) = 10$.
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