AoPS Online
Art of Problem Solving

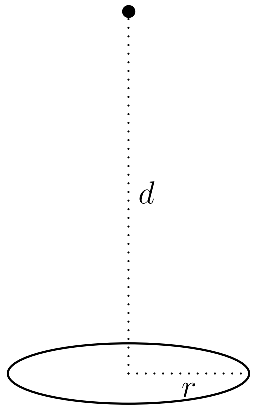
This is a test to determine whether you're familiar with the content of **Scholars High School Physics: Foundations**.

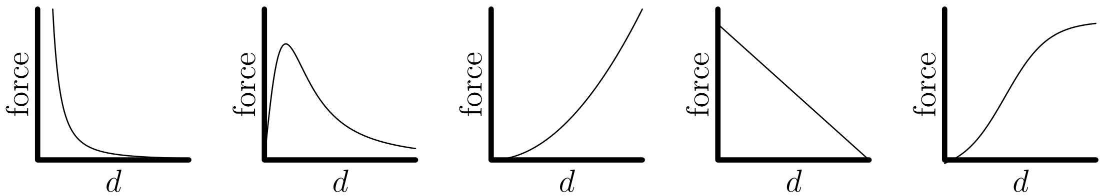
To use this test, we recommend:

- Take the test. You may use a calculator, but don't get other outside help.
- Have someone else check your responses using the solutions that start on page 11 of this document. They should only tell you whether your answer was correct or not.
- Take a second try on any problems you got wrong the first time.

If by the second try you can solve at least eight of the problems, then significant portions of the course would be a review for you.

1. Consider a circular ring with radius r and negligible thickness. A point particle sits at some distance d from the center of the ring on the symmetry axis of the ring. Which of the following is a plot of the magnitude of the gravitational force on the point particle from the ring as a function of d?





(a) (b) (c) (d) (e)

2. Which is an expression for the gravitational force on the point particle described in the previous problem? Let the total mass of the ring be M and the mass of the point particle be m.

   You may want to solve this problem via process of elimination.

   (a) $\dfrac{GMm}{d^2}$

   (b) $\dfrac{GMmd}{r^2 + d^3}$

   (c) $\dfrac{GMmd}{r^{3/2}}$

   (d) $\dfrac{GMmd^{3/2}}{(r^2 + d^2)^2}$

   (e) $\dfrac{GMmd}{(r^2 + d^2)^{3/2}}$

3. A team of scientists is considering four different hypotheses to explain a certain natural phenomenon. They design four different experiments, and write down the prediction of each hypothesis for each experiment. The predictions are discrete, and can be encoded as A, B, C, etc.

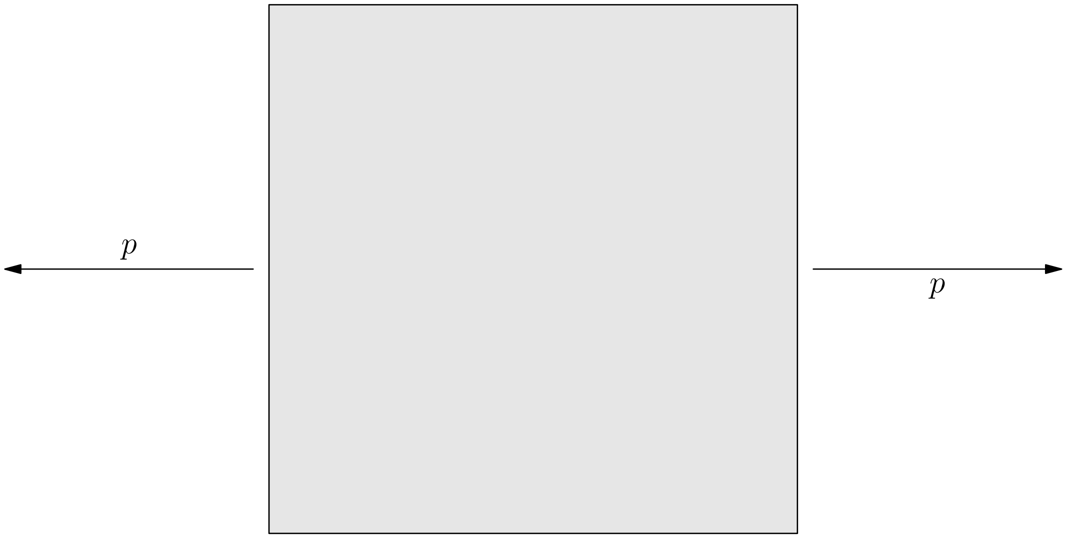
   Below is a chart showing the prediction of each hypothesis for each experiment.

	E1	E2	E3	E4
Hypothesis a	A	D	A	C
Hypothesis b	A	D	B	A
Hypothesis c	A	D	B	C
Hypothesis d	A	B	A	A

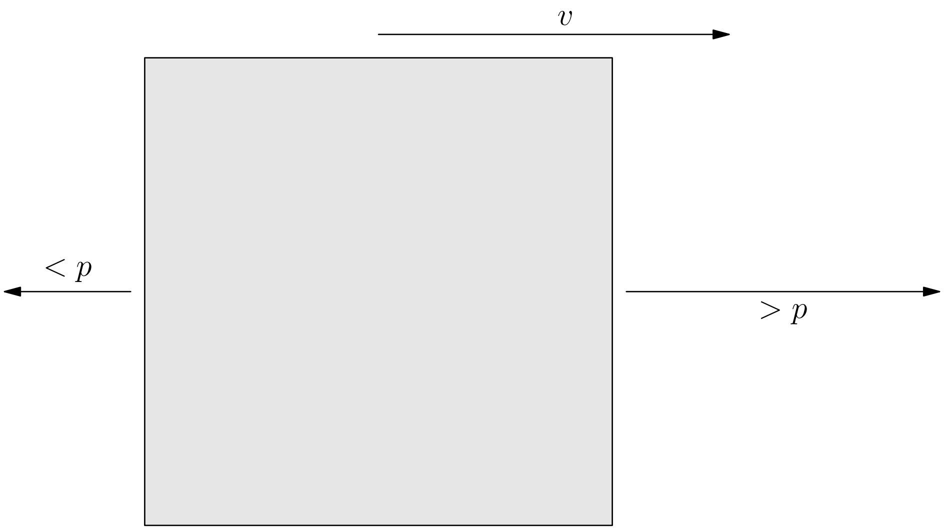
   Name exactly two experiments that the scientists could conduct in order to gain strong evidence that one particular hypothesis is the best of the four. Also name one experiment which would not give strong evidence for any hypothesis over the others.

   You may assume that at least one of the hypotheses makes correct predictions for all four experiments.

4. When first arguing for his famous formula $E = mc^2$, Albert Einstein conducted a thought experiment. He imagined a box that emitted two photons - one to the left and one to the right. The photons each carried the same magnitude of momentum and had the same frequency as each other. The situation is illustrated below, with p the magnitude of momentum.



Then Einstein considered the same event from a reference frame where the box initially moves to the right at speed v. In this reference frame, he knew that the photon emitted to the right would have higher frequency than the one emitted to the left due to Doppler shift, and so the photon emitted to the right would carry more momentum.



Briefly summarize what Einstein was able to conclude about the velocity of the box before, during, and after the emission of the photons, as viewed in this second reference frame.

5. In the SI unit system, the speed of light is set to $c = 299{,}792{,}458\ \mathrm{m} \cdot \mathrm{s}^{-1}$ by definition. This is used to define the length of the meter; there is an independent definition of the second.

   Consider a (fictional) different unit system, which we will call ”round number units”. Round number units use the second as a base unit of time, defined the same way as in the SI system. They also use the same kilogram as SI. However, round number units have a unit called one ”step” as the base unit of length. The step is defined by setting the speed of light to

$$c = 300{,}000{,}000\ \mathrm{step} \cdot \mathrm{s}^{-1}.$$

   We would like to compare the energy units between SI and round number units.

   In SI, the unit of energy is the joule, defined as

$$1\ \mathrm{J} = 1\ \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}.$$

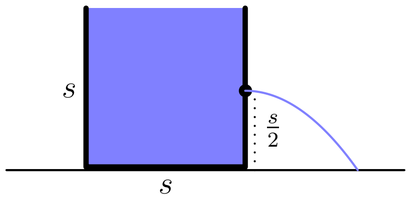
   In round number units, the unit of energy is the energon, defined as

$$1\ \mathrm{energon} = 1\ \mathrm{kg} \cdot \mathrm{step}^2 \cdot \mathrm{s}^{-2}.$$

   What is one joule of energy, measured in energons? Give your answer to at least five significant figures.

6. Shown below is a cube of side length s with an open top sitting on a frictionless surface. The cube is filled to the top with water. A small hole of area A is poked in the center of one side of the cube, allowing water to shoot out. At the moment the water begins shooting out of the hole, what is the speed of the water?

   You may assume that all the water leaves the hole with the same velocity, the cross-sectional area of the stream leaving the hole is A, and that frictional effects in the water can be ignored.



7. A small drop of liquid can oscillate, changing shape from a sphere to ellipsoid and back periodically. The period of this oscillation depends on the radius of the drop, its density, and its *surface tension*, a quantity related to the energy needed to create the drop from a large reservoir of liquid.

   These variables, along with their dimensions, are summarized in the table below.

Name of quantity	Symbol	Dimensional Formula
Period of oscillation	t	T
Surface tension	s	$\mathrm{M} \cdot \mathrm{T}^{-2}$
Density of liquid	d	$\mathrm{M} \cdot \mathrm{L}^{-3}$
Radius of drop	r	L

   The relationship between the period of oscillation and the radius of the drop is

   $$t \propto r^n$$

   for some number n. Find n.

8. The Cretaceous-Paleogene extinction event was a mass extinction that occurred about 65 million years ago. It was responsible for the extinction of most dinosaurs and many other animals.

   Many scientists believe this extinction occurred when a large asteroid hit the Earth. Dust from the impact is supposed to have blocked a significant amount of sunlight, leading to cold temperatures that killed many animals.

   Scientists have estimated via computer simulations that about one fifth of the mass of an asteroid ends up as dust in the atmosphere. For the asteroid from the Cretaceous-Paleogene extinction, this dust eventually created a nearly-uniform layer of $0.02\ \mathrm{g/cm^2}$ over the surface of the Earth. The density of most asteroids is approximately $2\ \mathrm{g/cm^3}$. The radius of Earth is about $6400\ \mathrm{km}$. Estimate the radius of the asteroid to the nearest kilometer.

9. Although there are different types of stars, all stars give off light in essentially the same way, called *blackbody radiation*. This phenomenon explains why anything that is hot will glow, giving off light. For example, molten iron and incandescent light bulbs glow simply because they are hot. Stars are also hot and glow in the same way.

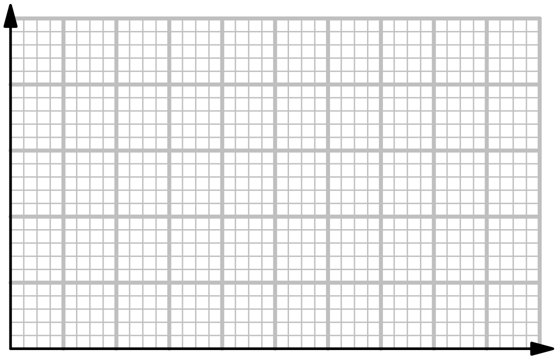
   Hotter objects glow brighter. To quantify this, we measure the power of the light radiation from the star - the energy carried away from the star by light per unit time. We will assume that any two stars at the same temperature will emit radiation at the same power per unit surface area. (In fact, this rule is not followed exactly, because while all objects give off black body radiation, they are not necessarily *perfect* black bodies. But stars are close enough to perfect black bodies for our purposes in this problem.) This assumption also means that if two stars are the same temperature, the larger one will emit more light.

   Below is a chart of stars, along with their radius, temperature, and the power of the radiation they emit.

star	radius (m)	Temperature (K)	Power (W)	
Sol (sun)	6.96×10^8	5778	3.85×10^{26}	
Alpha Centauri A	8.51×10^8	5790	5.84×10^{26}	
Sirius	1.19×10^9	9940	9.7×10^{27}	
Rigel	5.49×10^{10}	12,100	4.62×10^{31}	
Tau Ceti	5.52×10^8	5344	1.73×10^{26}	
10 Lacertae	5.75×10^9	36,000	3.92×10^{31}	
WR 136	3.55×10^9	70,800	2.31×10^{32}	
WR 142	5.57×10^8	200,000	3.51×10^{32}	
Betelgeuse	6.17×10^{11}	3500		

   Use the data, the empty columns of the chart, and the provided axes to find a model for the power emitted as a function of temperature and area for a star.

   Then estimate the power emitted by the red giant star Betelgeuse, whose temperature and radius are given in the last row of the table. Give your answer in watts, and use scientific notation. For example, an answer of 3 billion watts should be written "3×10^9 W."



Don't look at the next page until you've attempted all the problems!

Solutions

1. The correct choice is $\boxed{\text{(b)}}$.

   When $d = 0,$ the point particle is at the center of the ring. The gravitational force cannot have any component pointing left/right or forward/backward because the entire setup is symmetric about the axis. Additionally, there can be no up/down force, again due to symmetry. So at this point, the force is zero. This eliminates answers (a) and (d).

   When $d \gg r,$ the ring will span a vary small angle in the field of view of the point particle, and the gravitational field will be nearly the same as if the ring were also a point particle. This means the field should die away as d^{-2} for large $d,$ leaving only answer (b).

2. The correct choice is $\boxed{\text{(e)}}$.

   As in the previous problem, we conclude that the force should be zero when $d = 0.$ This eliminates answers (a) and (b).

   Next, we note that the force should fall away as d^{-2} when $d \gg r.$ This eliminates (c) and (d).

   Answer (b) adds r^2 to d^3. This is a unit mismatch, so this answer must be incorrect. The only remaining answer is (e).

3. The two experiments the scientists should conduct are experiments $\boxed{\text{3 and 4}}$.

   When the scientists conducts experiment 3, either hypotheses (a) and (d) will have correct predictions, or hypotheses (b) and (c) will. So the next experiment needs to make different predictions for (a) and (d) and for (b) and (c). Only experiment 4 does that.

   The experiment that will not give any good evidence in favor of one hypothesis over the others is $\boxed{\text{experiment 1}}$. All the hypotheses make the same prediction for that experiment, so no hypothesis has made a better prediction than any other.

4. Einstein concluded that $\boxed{\text{the velocity of the box was constant}}$ before, during, and after the photon emission. A correct answer should say that the velocity didn't change, that it was constant, or that it stayed at v before, during, and after the photons were emitted. If the answer doesn't explicitly state "before during, and after the photons were emitted", but does say that the velocity is constant or doesn't change, this should be counted as correct.

   Even though the two photons had different frequencies and carried different momentum, they did not affect the motion of the box. This is because in the original reference frame, the emission of photons was symmetric, so the motion of the box must have the same symmetry, meaning the box could not move at all.

   In the new reference frame, the box would still have zero acceleration, meaning its velocity wouldn't change.

   This may be counterintuitive because the photons have net momentum in this frame, meaning the rightward momentum of the box must decrease. It does this not by changing its velocity, but by losing mass.

5. $1\text{ J} \approx \boxed{1.0014\text{ energon}}$.

The speed of light is the same quantity, no matter the units, so

$$c = 299{,}792{,}458 \text{ m} \cdot \text{s}^{-1} = 300{,}000{,}000 \text{ step} \cdot \text{s}^{-1}.$$

Solving for one meter, we find

$$1 \text{ m} = \frac{300{,}000{,}000}{299{,}792{,}458} \text{ step}.$$

Next, we take the definition of the joule:

$$1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2},$$

and substitute in the previous equation for the meter:

$$1 \text{ J} = 1 \text{ kg} \cdot \left(\frac{300{,}000{,}000}{299{,}792{,}458}\text{step}\right)^2 \cdot \text{s}^{-2}.$$

Using the definition of the energon, this is

$$1 \text{ J} = \left(\frac{300{,}000{,}000}{299{,}792{,}458}\right)^2 \text{ energon} \approx 1.0014 \text{ energon}.$$

The answer can include more digits and still be counted as correct. If the student found

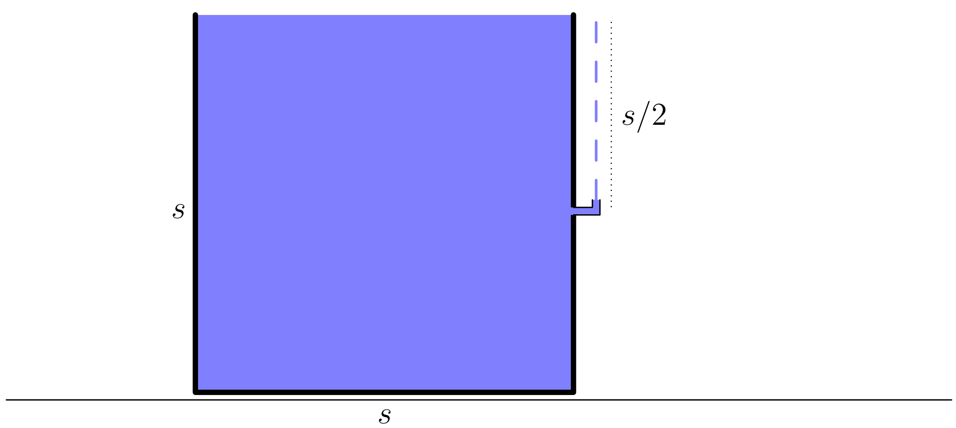
$$1 \text{ J} = \left(\frac{300{,}000{,}000}{299{,}792{,}458}\right)^2 \text{ energon}$$

and did not use a calculator to find the decimal expansion, that can also be counted as correct.

6. The speed is $\boxed{\sqrt{gs}}$.

   This question can be solved using conservation of energy.

   Suppose that, instead of a hole facing sideways, the water shot out of a spigot facing upwards.



Then the water would shoot upwards when coming out of the cube. It would have to shoot to exactly the top of the cube. If it shot any higher, we could catch the water, let it run downhill into the cube, and build a perpetual motion machine. But if it shot any lower, the reverse process would be a perpetual motion machine. Energy conservation says the water must shoot up a distance $s/2$.

So if a small mass m of water leaves at speed v, its kinetic energy would be

$$T = \frac{1}{2}mv^2,$$

and that would be just enough to meet the potential energy demand of rising a distance $s/2$,

$$U = \frac{s}{2}mg.$$

Setting $U = T$ and solving for v, we get

$$v = \sqrt{gs}.$$

As a second way to see this, we again imagine a small mass of water m shooting out of the hole. By energy conservation, its kinetic energy is equal to the potential energy lost by the water in the cube. The water in the cube below the hole is unchanged. The water above is also unchanged, except that a mass m is gone from the top, a distance $s/2$ above the hole. The potential energy released is therefore $mg \cdot s/2$. Setting this equal to the kinetic energy $\frac{1}{2}mv^2$, we rederive the same result.

7. The answer is $\boxed{n = 3/2}$ or $n = 1.5$.

   The time t must be some function of $s, d,$ and r. That function must have the dimension T because those are also the dimensions of t.

   Since, out of $s, d,$ and r, only s contains any time dimension, we start with the surface tension. If we take $s^{-1/2}$, we have

$$\left[s^{-1/2}\right] = \mathrm{M}^{-1/2} \cdot \mathrm{T}.$$

   This gives us the time dimension needed, but introduces an unwanted $\mathrm{M}^{-1/2}$ dimension. Of the remaining d and r, only d has any mass dimension. To cancel out the mass dimensions in s, we consider

$$\left[s^{-1/2}d^{1/2}\right] = \left(\mathrm{M}^{-1/2} \cdot \mathrm{T}\right) \cdot \left(\mathrm{M}^{1/2} \cdot \mathrm{L}^{-3/2}\right) = \mathrm{T} \cdot \mathrm{L}^{-3/2}.$$

   This cancels out the unwanted mass dimension, but introduces an unwanted length dimension. To cancel that out, we use r, the only remaining unused variable.

$$\left[s^{-1/2}d^{1/2}r^{3/2}\right] = \left(\mathrm{T} \cdot \mathrm{L}^{-3/2}\right) \cdot \mathrm{L}^{3/2} = \mathrm{T}.$$

   Finally, we have a quantity with dimensions T, so we conclude

$$t = c \cdot s^{-1/2}d^{1/2}r^{3/2}$$

   for some number c.

Regardless of c, we see that

$$t \propto r^{3/2}.$$

8. The radius was about $\boxed{4 \text{ km}}$.

   If the dust formed a uniform layer of 0.02 g/cm^2 using only 20% the mass of the asteroid, the entire asteroid would have left a layer five times as thick, or 0.1 g/cm^2.

   We can find the mass of the asteroid by estimating the total mass in such a layer. The surface area of Earth is

$$A \approx 4\pi \cdot (6.4 \times 10^8 \text{ cm})^2 \approx 5.1 \times 10^{18} \text{ cm}^2.$$

   That makes the mass of the asteroid

$$m \approx 5.1 \times 10^{18} \text{ cm}^2 \cdot 0.1 \text{ g/cm}^2 = 5.1 \times 10^{17} \text{ g}.$$

   Multiplying the volume of the asteroid to its density should give a second estimate of this mass, so the radius of the asteroid obeys

$$\frac{4}{3}\pi r^3 \cdot 2 \text{ g/cm}^3 \approx 5.1 \times 10^{17} \text{ g}.$$

   Solving for r, we find

$$r \approx 4 \times 10^5 \text{ cm} = 4 \text{ km}.$$

9. About $4.0 \times 10^{31} \text{ W}$.

   A correct answer should start with a number between 3.5 and 4.5. Then it should say "$\times 10^{31} \text{ W}$" next to it. (It must be 10^{31}, not 10^{30} or 10^{32}, etc.)

   For example, acceptable answers would be

   - $3.93 \times 10^{31} \text{ W}$
   - $4 \times 10^{31} \text{ W}$
   - $4.2 \times 10^{31} \text{ W}$.

   Unacceptable answers would be

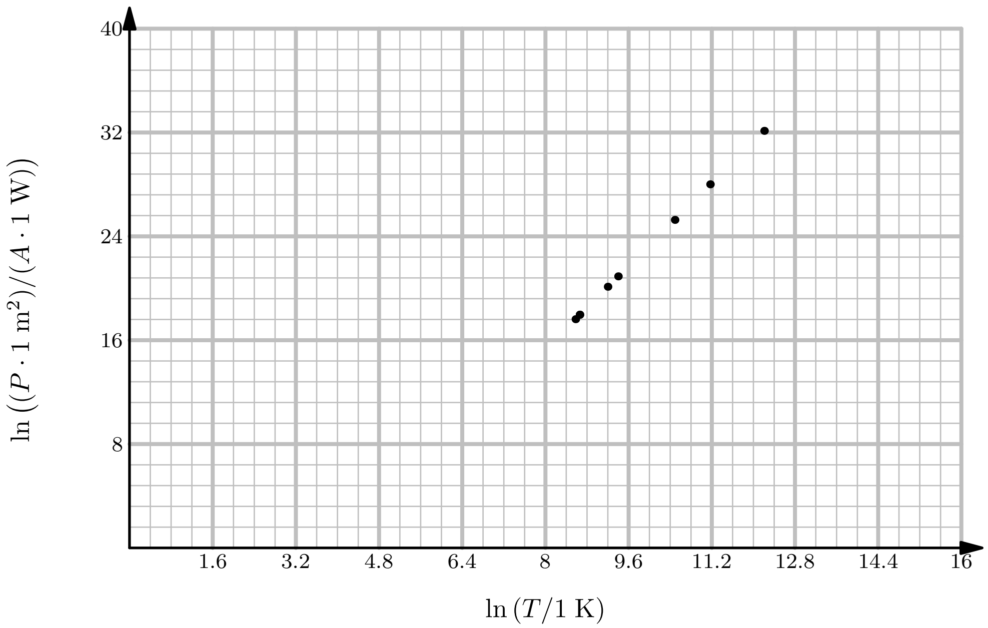
   - any answer that does not write "$\times 10^{31}$" after a number. e.g. $4.2 \times 10^{28} \text{ W}$ would be wrong
   - a number outside the range, e.g. $3.1 \times 10^{31} \text{ W}$, since 3.1 is too small

   If the student didn't include the W symbol, accept their answer as correct, but remind them that it's important to include this detail.

   To start this problem, we first note that we are interested in estimating the power per unit surface area of the star, so we fill out the last column of the chart with this quantity.

star	radius (m)	Temperature (K)	Power (W)	Power / Area (W/m^2)
Sol (sun)	6.96×10^8	5778	3.85×10^{26}	6.32×10^7
Alpha Centauri A	8.51×10^8	5790	5.84×10^{26}	6.42×10^7
Sirius	1.19×10^9	9940	9.7×10^{27}	5.48×10^8
Rigel	5.49×10^{10}	12,100	4.62×10^{31}	1.22×10^9
Tau Ceti	5.52×10^8	5344	1.73×10^{26}	4.52×10^7
10 Lacertae	5.75×10^9	36,000	3.92×10^{31}	9.44×10^{10}
WR 136	3.55×10^9	70,800	2.31×10^{32}	1.46×10^{12}
WR 142	5.57×10^8	200,000	3.51×10^{32}	9.00×10^{13}
Betelgeuse	6.17×10^{11}	3500		

We notice that the power per unit area increases with increasing temperature, and for the sun and alpha centauri, which have almost the same temperature, the power per unit area is almost the same. It appears that the power per unit area is a function of the temperature. We don't know what this function is, though. One way to find out is to plot our data on a log-log plot.



The plot is very nearly a straight line, indicating that the power per unit area is related to the temperature by a power law,

$$\frac{P}{A} = cT^{\alpha}.$$

The slope of this plot is almost exactly 4, so $\alpha = 4$, and we have

$$\frac{P}{A} = cT^{4}.$$

To find c, we can take any data point and plug it into the formula. We find $c = 2.27 \times 10^{-7} \frac{\mathrm{W}}{\mathrm{m}^2\ \mathrm{K}^4}$.

The power is therefore

$$\frac{P}{A} = 2.27 \times 10^{-7} T^4 \frac{\mathrm{W}}{\mathrm{m}^2\ \mathrm{K}^4}.$$

Plugging in the radius and temperature of Betelgeuse, we find an estimate for the power of $4 \times 10^{31}\ \mathrm{W}$. The best known figure is actually $4.85 \times 10^{31}\ \mathrm{W}$. The discrepancy is because Betelgeuse is not as close to an ideal blackbody as most other stars.